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Remarks to a paper of V. Komornik

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Let G be an arbitrary open interval on the real line, $q_1, q_2, \ldots, q_n \in L_1^{\mathrm{loc}}(G)$ arbitrary complex functions, and consider the formal differential operator

$$lu = u^{(n)} + q_1 u^{(n-1)} + \ldots + q_n u. \tag{1}$$

Let λ be an arbitrary complex number. The function $u \equiv 0$ is called an eigenfunction of order -1 with eigenvalue λ of the operator l. Assume that the eigenfunctions of order $m-1 \geqq -1$ are already defined, then a function $u\colon G \to \mathbf{C}$, $u \not\equiv 0$ is called an eingenfunction of order m with eigenvalue λ of the operator l, if the functions $u, u', \ldots, u^{(n-1)}$ are locally absolutely continuous on G and there exists an eigenfunction u^* of order $m-1$ of the operator l with the same eingenvalue such that a.e. on G

$$lu = \lambda u + u^*. \tag{2}$$

The eigenfunctions of higher order play important role in the theory of expansions [2]. Connected with this problem an upper estimate was given foı the sup-norm of the eigenfunctions in [1], on the basis of Titchmarsh formula. It was shown in [3] that this result is exact from the point of view of dependence on the eigenvalue. Later on V. Komornik [4] generalized the Titchmarsh formula for the differential operator (1) and using this result he obtained upper estimates for the eigenfunctions of (1), too.

The present paper has two purposes. First we give a new proof for Komornik's formula which is simpler than the former one and provides the coefficients in explicit form. Furthermore this approach sheds light on the inner beauty of this formula. As an application of this explicit expression we can prove exact lower estimates for the eigenfunctions of the operator (1).

For the sake of simplicity we consider only the operator

$$lu = u^{(n)}, \quad G = \mathbf{R}. \tag{3}$$

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The general case hence follows by the ideas of the paper [4] (by the variation of the constants).

1. Given a complex number $\lambda=\mu^n$ we denote by $S_k(\mu)$ the elementary symmetric polynomial of degree k of the variables $e^{\mu\omega_1}, e^{\mu\omega_2}, \dots, e^{\mu\omega_n}$ (where $\omega_1, \dots, \omega_n$ denotes the n-th roots of the unity) with the main coefficient $(-1)^k$, and

$$f_k(\mu) = (-1)^k S_{n-k}(e^{\mu\omega_1}, \dots, e^{\mu\omega_n}).$$

Obviously $f_k(\mu)=f_k(\mu\omega_1)=\dots=f_k(\mu\omega_n)$. Introduce also the functions $f_{0,k} \overset{\text{def}}{=} f_k$ $(k=0, 1, \dots, n)$,

$$f_{m,k} \overset{\text{def}}{=} \sum_{r=\max(0,k-mn)}^{\min(k,n)} f_r \cdot f_{m-1,k-r} \quad (m = 1, 2, \dots;\ k = 0, 1, \dots, (m+1)n).$$

Theorem 1. *Let u be an arbitrary eigenfunction of order $\leqq m$ of the operator (3) with some eigenvalue $\lambda=\mu^n$ $(m=0, 1, \dots)$. Then for any $x, t\in\mathbf{R}$,*

$$\sum_{k=0}^{(m+1)n} f_{m,k}(\mu t)\cdot u(x+kt) = 0. \tag{4}$$

Proof. First we show that

$$\hat{u}(x) \overset{\text{def}}{=} \sum_{k=0}^{n} f_k(\mu t)\cdot u(x+kt) \tag{5}$$

is an eigenfunction of order $\leqq m-1$ of the operator (3) with the eigenvalue λ. We consider only the case $\lambda\neq 0$ (the case $\lambda=0$ is similar). Then u has the form

$$u(x) = \sum_{p=1}^{n}\sum_{r=0}^{m} a_{pr} x^r e^{\mu\omega_p x}$$

with some constants a_{pr}. Then

$$\hat{u}(x) = \sum_{k=0}^{n} f_k(\mu t) \sum_{p=1}^{n}\sum_{r=0}^{m} a_{pr}(x+kt)^r e^{\mu\omega_p(x+kt)} =$$

$$= \sum_{k=0}^{n} f_k(\mu t) \sum_{p=1}^{n}\sum_{r=0}^{m} a_{pr} \sum_{s=0}^{r}\binom{r}{s} x^s (kt)^{r-s} e^{\mu\omega_p x} e^{k\mu\omega_p t} =$$

$$= \sum_{p=1}^{n}\sum_{s=0}^{m}\left[\sum_{r=s}^{m} a_{pr}\binom{r}{s}\sum_{k=0}^{n}(kt)^{r-s} f_k(\mu t) e^{k\mu\omega_p t}\right] x^s e^{\mu\omega_p x} = \sum_{p=1}^{n}\sum_{s=0}^{m} b_{ps} x^s e^{\mu\omega_p x},$$

where the numbers b_{ps} do not depend on x but depend on t. It suffices to show that $b_{1m}=\dots=b_{nm}=0$. But for any $1\leqq p\leqq n$,

$$b_{pm} = a_{pm}\prod_{q=1}^{n}(e^{\mu\omega_q t}-e^{\mu\omega_p t}) = 0.$$

Now we prove the formula (4) by induction on m. For $m=0$ it follows directly from the result just proved because the eigenfunctions of order -1 are identically zero. Assume (4) is valid for $m-1\geqq 0$. Then it is true also for m. Indeed, applying the induction hypothesis for $\hat{u}$ defined in (5) we obtain:

$$0=\sum_{k=0}^{mn} f_{m-1,k}(\mu t)\cdot\hat{u}(x+kt)=\sum_{k=0}^{mn} f_{m-1,k}(\mu t)\sum_{l=0}^{n} f_l(\mu t)u(x+kt+lt)=$$

$$=\sum_{r=0}^{(m+1)n}\left[\sum_{s=\max(0,r-mn)}^{\min(r,n)} f_s\cdot f_{m-1,r-s}(\mu t)\right]u(x+rt)=\sum_{r=0}^{(m+1)n} f_{m,r}(\mu t)u(x+rt).$$

Theorem 1 is proved.

2. We prove the following result.

Theorem 2. *Given any compact interval $K=[a,b]\subset\mathbf{R}$ there exists a positive constant $C=C(m)$ such that for any eigenfunction of order $m\geqq 0$ of the operator* (3) *with the eigenvalue λ*

$$\|u\|_{L^\infty(K)}\geqq C(1+|\mathrm{Re}\,\mu|)^{1/p}\|u\|_{L^p(K)}\quad(1\leqq p\leqq\infty). \tag{6}$$

Here μ denotes such an n-th root of λ for which $|\mathrm{Re}\,\mu|$ is minimal.

Proof. Given any $\lambda=\mu^n\in\mathbf{C}$ denote by $\mu_1,\ldots,\mu_n$ the n-th roots of λ such that $\mathrm{Re}\,\mu_1\geqq\ldots\geqq\mathrm{Re}\,\mu_n$. Obviously, putting $m=[n/2]$ or $m=[n/2]+1$, we have $\mathrm{Re}\,\mu_m\geqq 0$ and $\mathrm{Re}\,\mu_{m+1}\leqq 0$. We distinguish two cases: $|\mathrm{Re}\,\mu_m|\leqq|\mathrm{Re}\,\mu_{m+1}|$ or $|\mathrm{Re}\,\mu_m|>|\mathrm{Re}\,\mu_{m+1}|$ (the second case can occur only if n is odd). Let us consider first the case $|\mathrm{Re}\,\mu_m|\leqq|\mathrm{Re}\,\mu_{m+1}|$. Then obviously for any $t>0$

$$\left|\frac{f_k(\mu t)}{e^{(\mu_1+\ldots+\mu_{m-1})t}}\right|\leqq C\quad\text{if}\quad 0\leqq k\leqq n,\ n-k\neq m.$$

and

$$\left|\frac{f_{n-m}(\mu t)}{e^{(\mu_1+\ldots+\mu_{m-1})t}}-e^{\mu_m t}\right|\leqq C$$

where C is an absolute constant independent of μ and t. Hence, dividing the formula (4) by $e^{(\mu_1+\ldots+\mu_{m-1})t}$ we have for any $x\in R$ and $t>0$

$$|u(x)e^{\mu_m t}|\leqq C\sum_{k=0}^{n}|u(x-mt+kt)|\leqq$$

$$\leqq C\|u\|_{L^\infty(x-mt,x+(n-m)t)}\leqq C\|u\|_{L^\infty(x-nt,x+nt)}.$$

(Here and in the sequel C denotes an absolute constants not depending on the eigenfunction, not necessarily the same in different places.)

Now put $d(x)\overset{\mathrm{def}}{=}\min(x-a,b-x)$ for $x\in K$. Applying the above estimate we obtain for any $x\in K$ $(t\overset{\mathrm{def}}{=}d(x)/n)$

$$|u(x)e^{\mu_m d(x)/n}|\leqq C\|u\|_{L^\infty(K)},$$

whence

$$|u(x)| \leqq Ce^{-(\operatorname{Re}\mu_m)d(x)/n}\|u\|_{L^\infty(K)} \quad (\forall x\in K). \tag{7}$$

Consider now the case $|\operatorname{Re}\mu_m|>|\operatorname{Re}\mu_{m+1}|$. Then, for any $t<0$

$$\left|\frac{f_k(\mu t)}{e^{(\mu_{m+2}+\ldots+\mu_n)t}}\right| \leqq C \quad \text{if} \quad k\neq m+1$$

and

$$\left|\frac{f_{m+1}(\mu t)}{e^{(\mu_{m+2}+\ldots+\mu_n)t}} - e^{\mu_{m+1}t}\right| \leqq C,$$

Now we obtain for any $x\in R$ and $t<0$

$$|u(x)e^{\mu_{m+1}t}| \leqq C\|u\|_{L^\infty(x+nt,x-nt)}$$

and for any $x\in K$ $(t \overset{\text{def}}{=} -d(x)/n)$

$$|u(x)e^{-\mu_{m+1}d(x)/n}| \leqq C\|u\|_{L^\infty(K)},$$

whence

$$|u(x)| \leqq Ce^{(\operatorname{Re}\mu_{m+1})d(x)/n}\|u\|_{L^\infty(K)}. \tag{8}$$

Let us now introduce the notation

$$\varrho \overset{\text{def}}{=} (1/n)\min\{|\operatorname{Re}\mu_p|:\ 1\leqq p\leqq n\},$$

then in both cases

$$|u(x)| \leqq Ce^{-\varrho d(x)}\|u\|_{L^\infty(K)}. \tag{9}$$

Hence we proceed as in [3]: taking the $L^p(K)$-norm of both sides

$$\|u\|_{L^p(K)} \leqq C(2/p\varrho)^{1/p}\|u\|_{L^\infty(K)},$$

i.e.

$$\|u\|_{L^\infty(K)} \geqq C^p\sqrt[p]{\varrho}\,\|u\|_{L^p(K)}.$$

Hence (6) follows for $\varrho\geqq 1$. On the other hand, the case $\varrho<1$ is trivial and Theorem 2 is proved for $m=0$. The general case follows by induction on m.

References

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